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Unitary Whitehead Group of Cyclic Groups

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1. INTRODUCTION

In topology, one encounters certain groups known as surgery obstruction groups, introduced by C. T. C. Wall [10]. They can be described in a purely algebraic setting, and so hopefully be computed by algebraic means, notably by techniques developed mainly by H. Bass [2], which has come to be known as algebraic K-theory. This paper aims at applying such techniques to the computation of the so-called unitary Whitehead groups, certain quotients of which are the Wall's surgery groups mentioned above.

We shall recall certain basic notions about quadratic forms in Section 2. In the same section we introduce the unitary Whitehead group as the Whitehead group of a certain category, then exhibit it as the commutator quotient group of a certain matrix group. Both descriptions are useful in the sequel, and we shall not hesitate in switching from one to the other. In Section 3 we introduce the necessary machinery for the computation, those being the Mayer-Vietoris sequence associated to a Cartesian square and the unitary analogue of Milnor's K_2 group. Then Section 4 paves the way for the actual computation with the crux of the work done in Section 5 and the round-up in Section 6.[1]

This paper forms part of Chapter I of the author's dissertation [8] written under Professor Hyman Bass. I am deeply indebted to him for his extraordinary patience, generous help and inspiring guidance, which made my years at Columbia University an extremely valuable and pleasurable experience.

[1] After this paper was written up, the author learned from A. Bak by oral communication that he has treated similar questions, by rather different methods. He announced some results which overlap substantially with those presented here, but the author has not seen the proofs yet.

2. Basic Notions

A unitary ring is a triple (A, λ, Λ), where A is a ring with involution denoted by $a \mapsto \bar{a}$, λ is an element in the centre of A satisfying $\lambda\bar{\lambda} = 1$, and Λ is an additive subgroup of A satisfying the conditions

$$S_{-\lambda}(A) = \{a - \lambda\bar{a} \mid a \in A\} \subset \Lambda \subset \{a \in A \mid a + \lambda\bar{a} = 0\} = S^{-\lambda}(A),$$

and $\bar{a}ra \in \Lambda$ whenever $a \in A$, $r \in \Lambda$. This notion of a "sliding parameter group Λ" is first introduced by A. Bak in his thesis [1]. A morphism $f: (A, \lambda, \Lambda) \to (A', \lambda', \Lambda')$ between two unitary rings is a ring homomorphism $f: A \to A'$ satisfying the conditions $f(\lambda) = \lambda'$, $f(\bar{a}) = \overline{f(a)}$ for all $a \in A$ and $f(\Lambda) \subset \Lambda'$. By an epimorphism of unitary rings, we mean a morphism with $f(A) = A'$ and $f(\Lambda) = \Lambda'$. When A has trivial involution (that is $a = \bar{a}$ for all $a \in A$, so that A has to be commutative), we single out the following three cases and designate them by special names: (1) The case $(A, 1, 0)$ is called the *orthogonal case.* (2) The case $(A, -1, A)$ is called the *(full) sympletic case.* (3) The case $(A, -1, \Lambda)$ is called the *restricted sympletic case.*

Suppose M is a right A-module. A sesquilinear form on M is a biadditive map $B: M \times M \to A$ satisfying the condition $B(xa, yb) = \bar{a}B(x, y)b$. The set of all sesquilinear forms on M, denoted $\mathrm{Sesq}_A(M)$, form a group under the addition given by $(B_1 + B_2)(x, y) = B_1(x, y) + B_2(x, y)$ for $x, y \in M$. We define a group homomorphism $T_\lambda : \mathrm{Sesq}_A(M) \to \mathrm{Sesq}_A(M)$ by the formula $(T_\lambda B)(x, y) = \lambda\overline{B(y, x)}$ for $x, y \in M$. The group $H_\lambda(M) = \ker (I - T_\lambda)$ is called the group of all λ-hermitian forms on M. The group $Q_\lambda(M) = \mathrm{coker}\,(I - T_\lambda)$ is called the group of all λ-quadratic forms on M. The quotient map $\mathrm{Sesq}_A(M) \to Q_\lambda(M)$ will be denoted by $B \mapsto [B]$. Since $T_\lambda^2 = I$, the association $[B] \mapsto (I + T_\lambda)B = \langle\ ,\ \rangle_{[B]}$ is well-defined from $Q_\lambda(M)$ into $H_\lambda(M)$. The association $x \mapsto q_{[B]}(x)$, where $q_{[B]}(x) = B(x, x) \bmod \Lambda$, is also well-defined from M into A/Λ.

We take only finitely generated projective right A-modules P and only non-singular λ-quadratic forms on P (that is, the map $x \mapsto \langle x, -\rangle_{[B]}$ is an isomorphism from P onto $P^* = \mathrm{Hom}_A(P, A)$, where P^* is viewed as a right A-module via the involution on A). Such pairs $(P, [B])$ form the objects of a category in which a morphism $\sigma : (P, [B]) \to (Q, [C])$ is an A-isomophism $\sigma : P \to Q$ satisfying $\langle \sigma x, \sigma y\rangle_{[C]} = \langle x, y\rangle_{[B]}$ and $q_{[C]}(\sigma x) = q_{[B]}(x)$. We shall denote this new category by $Q(A, \lambda, \Lambda)$. It has a product (in the sense of Bass [2, Chapter VII]), namely the orthogonal sum $(P, [B]) \perp (Q, [C]) = (P \oplus Q, [B \oplus C])$ where $(B \oplus C)((x_1, y_1), (x_2, y_2)) = B(x_1, x_2) + C(y_1, y_2)$ for $x_1, x_2 \in P$ and $y_1, y_2 \in Q$.

An important example of quadratic form is afforded by the so-called hyperbolic form. Suppose P is a finitely generated projective right A-module.

We define a sesquilinear form on $P \oplus P^*$ by $B_P((x,f),(y,g)) = f(y)$ for $x, y \in P$, and $f, g \in P^*$. It can be shown that $[B_P]$ is nonsingular [5, Chapter I, (4.7)], so that $H(P) = (P \oplus P^*, [B_P])$ is an object of the category $Q(A, \lambda, \Lambda)$. The association $P \mapsto H(P)$ and $\alpha \mapsto H(\alpha) = \alpha \oplus \alpha^{*-1}$ define a functor $H : \mathscr{P}(A) \to Q(A, \lambda, \Lambda)$, where $\mathscr{P}(A)$ is the category of finitely generated projective right A-modules and A-isomorphisms. The functor H is product preserving and cofinal (in the sense of Bass [2, Chapter VII]) [5, Chapter I, (4.7), (4.8)], so that it induces a homomorphism $H : K_1(A) \to KU_1^{\lambda}(A, \Lambda)$, where we write $KU_1^{\lambda}(A, \Lambda)$ in place of $K_1(Q(A, \lambda, \Lambda))$, and call it the unitary Whitehead group of the ring A. The cokernel of the homomorphism H will be denoted by $W_1^{\lambda}(A, \Lambda)$.

By general argument, it can be shown [8, Appendix, (4.2)] that $KU_1^{\lambda}(A, \Lambda) = U^{\lambda}(A, \Lambda)/[U^{\lambda}(A, \Lambda), U^{\lambda}(A, \Lambda)]$ where $U^{\lambda}(A, \Lambda)$ is the group inj lim $U_{2n}^{\lambda}(A, \Lambda)$, with $U_{2n}^{\lambda}(A, \Lambda)$ being the group of $2n \times 2n$ invertible matrices $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ satisfying the conditions $\sigma^{-1} = \left(\begin{smallmatrix} \delta^* & \lambda\beta^* \\ \bar{\lambda}\gamma^* & \alpha^* \end{smallmatrix}\right)$ and diagonal entries of $\beta\alpha^*$, $\delta\gamma^*$ in Λ, the filtering being given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Vasserstein has shown (see [5, Chapter II, (5.2)] that $[U^{\lambda}(A, \Lambda), U^{\lambda}(A, \Lambda)] = EU^{\lambda}(A, \Lambda)$, the subgroup of $U^{\lambda}(A, \Lambda)$ generated by the matrices

$$X_+(\beta) = \begin{pmatrix} I & \beta \\ 0 & I \end{pmatrix} \quad \text{and} \quad X_-(\gamma) = \begin{pmatrix} I & 0 \\ \gamma & I \end{pmatrix}.$$

In this language, the homomorphism H is given by $H(\alpha) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{smallmatrix}\right)$.

In what follows, we are interested in the case $A = \mathbf{Z}\pi$, the integral group ring of the group π, with involution given by $g \mapsto \bar{g} = g^{-1}$ for all $g \in \pi$, and $\lambda = 1$ or -1, and $\Lambda = S_{-\lambda}(\mathbf{Z}\pi)$. Our problem can now be stated in one simple sentence: Compute the groups $KU_1^{\lambda}(\mathbf{Z}\pi, \Lambda)$ and $W_1^{\lambda}(\mathbf{Z}\pi, \Lambda)$. We shall denote $L_1(\pi) = W_1^{1}(\mathbf{Z}\pi, S_{-1}(\mathbf{Z}\pi))/\langle w_1 \rangle$ where w_1 is represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We shall denote $L_3(\pi) = W_1^{-1}(\mathbf{Z}\pi, S_1(\mathbf{Z}\pi))/\langle w_{-1} \rangle$ where w_{-1} is represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The case $L_3(\pi)$ has been extensively investigated by Bass [3], starting from a computation by Lee [7] of the case π cyclic of odd prime order. The main result is $L_3(\pi) = 0$ for π finite abelian of *odd* order. In this paper we shall show that $L_1(\pi) = 0$ for π cyclic of odd prime power order. I would conjecture that $L_1(\pi) = 0$ for π cyclic of odd order.

3. Mayer-Vietoris Sequence

In this section we are going to define a unitary analogue of Milnor's K_2 group, then string together the so-defined $KU_2{}^\lambda$ and $KU_1{}^\lambda$ groups into an exact sequence, and finally comment on what we can say about such $KU_2{}^\lambda$ groups in the orthogonal case.

Bass has shown [5, Chapter II, (5.2)] that the group $EU^\lambda(A, \Lambda)$ is perfect, so that it possesses a universal central covering, which is denoted by $\widetilde{EU}^\lambda(A, \Lambda)$. The kernel of this universal central covering will be named $KU_2{}^\lambda(A, \Lambda)$. It can be seen that $KU_2{}^\lambda$ is a covariant functor from the category of rings to the category of abelian groups. It is sometimes preferrable to have a categorical definition for $KU_2{}^\lambda$. Bass has achieved this by giving a categorical definition for K_2 of any category with product [5, Chapter III, (A.3)] and showing that the two definitions coincide [5, Chapter III, (A.6)]. The next task is to connect up such $KU_2{}^\lambda$ groups with the $KU_1{}^\lambda$ groups.

Theorem 3.1. *Suppose*

$$\begin{tikzcd} (A, \Lambda) \arrow[r, "i_1"] \arrow[d, "i_2"'] & (A_1, \Lambda_1) \\ (A_2, \Lambda_2) \arrow[r, "j_1"] & (A', \Lambda') \arrow[u, "j_2"'] \end{tikzcd} \tag{3.2}$$

is a cartesian square of unitary rings (λ is the same throughout), with all i_1, i_2, j_1, j_2 being epimorphisms of unitary rings. Then there is an exact sequence

$$KU_2{}^\lambda(A_1, \Lambda_1) \oplus KU_2{}^\lambda(A_2, \Lambda_2) \xrightarrow{j_2 - j_1} KU_2{}^\lambda(A', \Lambda') \xrightarrow{\partial} KU_1{}^\lambda(A, \Lambda)$$
$$\xrightarrow{(i_1, i_2)} KU_1{}^\lambda(A_1, \Lambda_1) \oplus KU_1{}^\lambda(A_2, \Lambda_2) \xrightarrow{j_2 - j_1} KU_1{}^\lambda(A', \Lambda') \tag{3.3}$$

Remark 3.4. By a cartesian square of unitary rings, we mean that the square of rings as well as the square of Λ-groups are both cartesian. The exact sequence (3.3) is called the Mayer-Vietoris sequence associated to the cartesian square (3.2).

Proof. See [5, Chapter III, (2.3)].

The group $KU_2{}^\lambda(A, \Lambda)$ is in general fairly difficult to handle. The situation is somewhat better in the orthogonal case, thanks to results of Bass, Matsumoto, Stein and Steinberg. The central notions are that of the group $\mathrm{Epin}(A)$ and $\mathrm{KSpin}_2(A)$ of a ring A. In the orthogonal case, we shall write KO_1, KO_2, EO instead of $KU_1{}^1$, $KU_2{}^1$, EU^1 respectively.

THEOREM 3.5 (Bass). *Suppose A is a commutative ring with trivial involution. Then there is a perfect group* $\mathrm{Epin}(A)$ *which fits into an exact sequence*

$$1 \to \mu_2(A) \to \mathrm{Epin}(A) \to EO(A) \to 1, \tag{3.6}$$

where $\mu_2(A)$, the multiplicative group of square roots of unity in A, is central in $\mathrm{Epin}(A)$. Furthermore the sequence (3.6) *is natural with respect to A.*

Proof. See [4, (4.3.4)].

The group $\mathrm{Epin}(A)$ possesses a universal central covering, which is denoted by $\widetilde{\mathrm{Epin}}(A)$. The kernel of this universal central covering will be named $\mathrm{KSpin}_2(A)$.

THEOREM 3.7. *Suppose A is a commutative ring with trivial involution. Then there is an exact sequence*

$$0 \to \mathrm{KSpin}_2(A) \to KO_2(A) \to \mu_2(A) \to 0. \tag{3.8}$$

Furthermore the sequence (3.8) *is natural with respect to A.*

Proof. Since $\mathrm{Epin}(A) \to EO(A)$ is a central covering, it induces an isomorphism $\widetilde{\mathrm{Epin}}(A) \xrightarrow{\cong} \widetilde{EO}(A)$ by general argument about central coverings. Hence there is the following commutative diagram with exact rows, whose kernels and cokernels fit into the (dotted) exact sequence, which is (3.8).

$$\begin{tikzcd}
 & 0 \arrow[r, dashed] \arrow[d] & \mathrm{KSpin}_2(A) \arrow[r, dashed] \arrow[d] & KO_2(A) \arrow[d] \arrow[dddll, dashed] & \\
 & 1 \arrow[r] \arrow[d] & \widetilde{\mathrm{Epin}}(A) \arrow[r, "\cong"] \arrow[d] & \widetilde{EO}(A) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_2(A) \arrow[r] \arrow[d] & \mathrm{Epin}(A) \arrow[r] \arrow[d] & EO(A) & \\
 & \mu_2(A) \arrow[r, dashed] & 0 & &
\end{tikzcd}$$

4. SET-UP OF COMPUTATION

From now on, π will denote a finite abelian group of order m. We take its integral group ring $\mathbf{Z}\pi$ and put $\Sigma = \sum_{g \in \pi} g \in \mathbf{Z}\pi$. Since $\bar{\Sigma} = \Sigma$, the ideal (Σ) generated by Σ is involutary. So the involution on $\mathbf{Z}\pi$ induces an involution on the quotient $\mathbf{Z}\pi/(\Sigma)$, hereafter denoted by $\mathbf{Z}\pi^*$, such that $\overline{x + (\Sigma)} = \bar{x} + (\Sigma)$ for all $x \in \mathbf{Z}\pi$. Because $g\Sigma = \Sigma$ for all $g \in \pi$, every element in (Σ) can be written simply as $n\Sigma$ with $n \in \mathbf{Z}$.

It is not hard to see [8, Chapter I, (6.5)] that the square

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{i_1} & \mathbf{Z}\pi^* \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_2} \\ \mathbf{Z} & \xrightarrow{j_1} & \mathbf{Z}/m\mathbf{Z} \end{array}$$

is Cartesian, where i_1, j_1 are the natural epimorphisms, i_2 is the augmentation map, and j_2 maps $x + (\Sigma)$ to $i_2(x) + m\mathbf{Z}$. To say a bit more, we have the following result.

THEOREM 4.1. *The following is a Cartesian square of unitary rings, with all homomorphisms being epimorphisms (of unitary rings). Here i_1, i_2, j_1, j_2 are as described above, $\lambda = 1$ throughout, and $\Lambda = S_{-1}(\mathbf{Z}\pi)$, $\Lambda^* = S_{-1}(\mathbf{Z}\pi^*)$,*

$$\begin{array}{ccc} (\mathbf{Z}\pi, \Lambda) & \xrightarrow{i_1} & (\mathbf{Z}\pi^*, \Lambda^*) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_2} \\ (\mathbf{Z}, 0) & \xrightarrow{j_1} & (\mathbf{Z}/m\mathbf{Z}, 0). \end{array} \tag{4.2}$$

Proof. Since all i_1, i_2, j_1, j_2 are surjective and involution-preserving, so $i_1(\Lambda) = \Lambda^*$, $i_2(\Lambda) = 0$, $j_1(0) = 0$ and $j_2(\Lambda^*) = 0$. We know the square of rings is cartesian. To see that the corresponding square of Λ's is also Cartesian, it suffices to check that one of j_1 or j_2 is surjective on the fixed ring of involution [8, Appendix, (5.5)]. Indeed, j_1 is so because the fixed rings are $\mathbf{Z}$ and $\mathbf{Z}/m\mathbf{Z}$ respectively.

THEOREM 4.3. *Same notation as in Theorem* (4.1), *the sequence*

$$KU_2^1(\mathbf{Z}\pi^*, \Lambda^*) \oplus KO_2(\mathbf{Z}) \xrightarrow{j_2 - j_1} KO_2(\mathbf{Z}/m\mathbf{Z}) \xrightarrow{\partial} KU_1^1(\mathbf{Z}\pi, \Lambda)$$
$$\xrightarrow{(i_1, i_2)} KU_1^1(\mathbf{Z}\pi^*, \Lambda^*) \oplus KO_1(\mathbf{Z}) \xrightarrow{j_2 - j_1} KO_1(\mathbf{Z}/m\mathbf{Z}) \tag{4.4}$$

is exact.

Proof. The sequence (4.4) is the Mayer-Vietoris sequence associated to the cartesian square (4.2).

If π is cyclic, say with generator t, then by putting τ to be the image of t modulo (Σ), we may view $\mathbf{Z}\pi^*$ as the ring $\mathbf{Z}[\tau]$ where $1 + \tau + \cdots + \tau^{m-1} = 0$. In this case, the homomorphisms can be written more explicitly as follows:

$$i_1(t) = \tau, \quad i_2(t) = 1, \quad j_1(1) = 1 + m\mathbf{Z}, \quad j_2(\tau) = 1 + m\mathbf{Z}.$$

Further assume m is *odd*, then $\mathbf{Z}\pi^*$ satisfies the so-called "evenness condition", that is $1 = e + \bar{e}$ for some $e \in \mathbf{Z}\pi^*$, namely take e to be the element $-\tau - \cdots - \tau^{(m-1)/2}$.

5. Computation on $KU_1^1(\mathbf{Z}\pi^*, A^*)$

This section is perhaps the crucial one in the whole computation. The key results are Theorem (5.5), Theorem (5.6) and Theorem (5.8). The idea in Theorem (5.5) is a variation of that of R. Lee [7]. We shall begin by describing certain groups of an "ad-hoc" nature. For this, we shall work in a general setting.

Let A be a ring with involution, and q be a two-sided involutary ideal in A. The group $T_+U_{2n}^\lambda(A; q)$ consist of all $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in U_{2n}^\lambda(A)$ such that $\beta \equiv 0 \bmod q$. The group $T_-U_{2n}^\lambda(A; q)$ consists of all $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in U_{2n}^\lambda(A)$ such that $\gamma \equiv 0 \bmod q$. Let $ET_+U_{2n}^\lambda(A; q)$ denote the subgroup of $T_+U_{2n}^\lambda(A; q)$ generated by the special type of matrices $X_+(\beta)$ where $\beta \equiv 0 \bmod q$, $X_-(\gamma)$, and $H(\epsilon)$ where $\epsilon \in E_n(A)$. Let $ET_-U_{2n}^\lambda(A; q)$ denote the subgroup of $T_-U_{2n}^\lambda(A; q)$ generated by the special type of matrices $X_+(\beta)$, $X_-(\gamma)$ where $\gamma \equiv 0 \bmod q$, and $H(\epsilon)$ where $\epsilon \in E_n(A)$. As usual, we have filterings

$$T_\pm U_2^\lambda(A; q) \hookrightarrow T_\pm U_4^\lambda(A; q) \hookrightarrow T_\pm U_6^\lambda(A; q) \hookrightarrow \ldots,$$

and

$$ET_\pm U_2^\lambda(A; q) \hookrightarrow ET_\pm U_4^\lambda(A; q) \hookrightarrow ET_\pm U_6^\lambda(A; q) \hookrightarrow \ldots .$$

We shall define $T_\pm U^\lambda(A; q) = \text{inj lim } T_\pm U_{2n}^\lambda(A; q)$ and

$$ET_\pm U^\lambda(A; q) = \text{inj lim } ET_\pm U_{2n}^\lambda(A; q).$$

Proposition 5.1. $[T_\pm U^\lambda(A; q), T_\pm U^\lambda(A; q)] \subset ET_\pm U^\lambda(A; q)$.

Proof. For convenience, we write TU_{2n} and ETU_{2n} for $T_\pm U_{2n}^\lambda(A; q)$ and $ET_\pm U_{2n}^\lambda(A; q)$ respectively. Given $\sigma = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ and $\sigma' = \left(\begin{smallmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{smallmatrix}\right)$ in TU_{2n}, $\sigma \perp \sigma'$ will denote the matrix

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha' & 0 & \beta' \\ \gamma & 0 & \delta & 0 \\ 0 & \gamma' & 0 & \delta' \end{pmatrix}$$

in TU_{4n}. By the following formula

$$\sigma \perp \sigma^{-1} = X_-\begin{pmatrix} 0 & \gamma \\ -\bar{\lambda}\gamma^* & \bar{\lambda}\gamma^*\alpha \end{pmatrix} H\begin{pmatrix} I_n & \alpha \\ 0 & I_n \end{pmatrix} H\begin{pmatrix} I_n & 0 \\ -\delta^* & I_n \end{pmatrix} X_+\begin{pmatrix} 0 & \beta \\ -\lambda\beta^* & \delta^*\beta \end{pmatrix}$$

$$\times X_-\begin{pmatrix} 0 & \gamma \\ -\bar{\lambda}\gamma^* & \bar{\lambda}\gamma^*\alpha \end{pmatrix} H\begin{pmatrix} I_n & \alpha \\ 0 & I_n \end{pmatrix} H\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

we see that $\sigma \perp \sigma^{-1} \in ETU_{4n}$ for all $\sigma \in TU_{2n}$. Since $[\sigma, \sigma'] \perp I_{2n} = [(\sigma\sigma') \perp (\sigma\sigma')^{-1}]\,[\sigma^{-1} \perp \sigma]\,[\sigma'^{-1} \perp \sigma']$, so $[\sigma, \sigma'] \perp I_{2n}$ is in ETU_{4n} for all σ, σ' in TU_{2n}. Going to the direct limit, this says what we set out to prove.

Hence $ET_\pm U^\lambda(A; q)$ is a normal subgroup of $T_\pm U^\lambda(A; q)$ and the quotient $T_\pm U^\lambda(A; q)/ET_\pm U^\lambda(A; q)$ is abelian. We shall denote this quotient by $KT_\pm U_1{}^\lambda(A; q)$. By definition we have $H(E(A)) \subset ET_\pm U^\lambda(A; q)$, so that H induces a homomorphism $H : K_1(A) \to KT_\pm U_1{}^\lambda(A; q)$. Its cokernel will be denoted by $T_\pm W_1{}^\lambda(A; q)$.

We now make the following standing hypothesis, to be agreed upon throughout this section. A is a commutative ring with involution $a \mapsto \bar{a}$. $\lambda = \pm 1$ as the case may be, but will be indicated clearly. A satisfies the "evenness condition" (see the end of Section 4), one consequence of which is that A has to be the smallest choice. A_0 denotes the fixed ring of A under involution. u denotes a nonzero-divisor in A satisfying the condition $u + \bar{u} = 0$. Notice that the ideal (u) generated by u is involutary. The "eveness condition" on A also guarantees that for a given $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in T_+U_{2n}^{-1}(A; (u))$, we can arrange to have $\beta\alpha^* = u\epsilon + (u\epsilon)^*$ for some $\epsilon \in M_n(A)$ (we already know that $\beta\alpha^* = \nu + \nu^*$ for some $\nu \in M_n(A)$) [8, Chapter I, Section 8].

THEOREM 5.2. *The map*

$$\begin{pmatrix} \alpha & u\beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ u\gamma & \delta \end{pmatrix}$$

defines an isomorphism $\Phi : T_+U^{-1}(A; (u)) \to T_-U^1(A; (u))$, *which induces isomorphisms* $\Phi : KT_+U_1^{-1}(A; (u)) \to KT_-U_1{}^1(A; (u))$ *and* $\Phi : T_+W_1^{-1}(A; (u)) \to T_-W_1{}^1(A; (u))$.

Proof. We first check that Φ indeed maps into $T_-U^1(A; (u))$, which means we have to show

$$\begin{pmatrix} \alpha & \beta \\ u\gamma & \delta \end{pmatrix} \in U^{-1}(A).$$

Since

$$\begin{pmatrix} \alpha & u\beta \\ \gamma & \delta \end{pmatrix} \in T_+U^{-1}(A; (u)),$$

we have $\alpha\delta^* - (u\beta)\gamma^* = I$, $(u\beta)\alpha^* = u\epsilon + (u\epsilon)^*$ for some ϵ, $\delta\gamma^* = \eta + \eta^*$ for some η. Hence, $\alpha\delta^* + \beta(u\gamma)^* = \alpha\delta^* - (u\beta)\gamma^* = I$, $\beta\alpha^* = \epsilon - \epsilon^*$ since $u\beta\alpha^* = u\epsilon - u\epsilon^*$ and u is a non-zero-divisor, $\delta(u\gamma)^* = u(\delta\gamma^*) = u(\eta + \eta^*) = u\eta - (u\eta)^*$. This says what we want. Clearly Φ is a homomorphism. It has an inverse given by

$$\Psi: \begin{pmatrix} \alpha & \beta \\ u\gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & u\beta \\ \gamma & \delta \end{pmatrix}$$

(by a similar checking). Next we check that $\Phi(ET_+U^{-1}(A; (u)) = ET_-U^1(A; (u))$, which would give the last two assertions. Suffices to check that

$$\Phi(ET_+U^{-1}(A; (u)) \subset ET_-U^1(A; (u))$$

and

$$\Psi(ET_-U^1(A; (u)) \subset ET_+U^{-1}(A; (u)).$$

We only do the first inclusion, the second being similar. Suffices to show $\Phi(\sigma) \in EU^1(A)$ where σ is of the form $X_+(\beta)$ where $\beta \equiv 0 \bmod (u)$, $X_-(\gamma)$, or $H(\epsilon)$ where $\epsilon \in E(A)$. But this is clear in either case.

Remark 5.3. The composites

$$KT_+U_1^{-1}(A; (u)) \to KT_-U_1^1(A; (u)) \to KU_1^1(A)$$

and

$$T_+W_1^{-1}(A; (u)) \to T_-W_1^1(A; (u)) \to W_1^1(A)$$

will play a key role in Theorem 5.6. Here, the first arrow is Φ, and the second arrow is that induced by $T_-U^1(A; (u)) \subset U^1(A)$. We shall still denote these composites by Φ. Hereafter, $KT_+U_1^{-1}(A; (u))$ and $T_+W_1^{-1}(A; (u))$ will simply be denoted by $KTU_1(A; (u))$ and $TW_1(A; (u))$ respectively. Also, $KT_+U_1^{-1}(A/(u); (0))$ and $T_+W_1^{-1}(A/(u); (0))$ will simply be denoted by $KTU_1(A/(u))$ and $TW_1(A/(u))$ respectively. Our main concern below is to see when is $TW_1(A; (u)) = 0$.

PROPOSITION 5.4. $TW_1(A/(u)) = 0$.

Proof. The element $\left(\begin{smallmatrix} \alpha & 0 \\ \gamma & \delta \end{smallmatrix}\right) \in TU(A/(u))$ implies $\alpha\delta^* = I$ and $\delta\gamma^* = \eta + \eta^*$ for some η. Hence $\alpha^{-1} = \delta^*$ and $(\gamma\alpha^{-1})^* = (\alpha^{-1})^*\gamma^* = \delta\gamma^* = \eta + \eta^*$. The formula

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} I & 0 \\ \gamma\alpha^{-1} & I \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}$$

says that $\left(\begin{smallmatrix} \alpha & 0 \\ \gamma & \delta \end{smallmatrix}\right)$ represents zero in the group $TW_1(A/(u))$.

THEOREM 5.5. *Suppose*

(*a*) *the composite* $(1+(u))^{\bullet} \xrightarrow{H} KU_1^{-1}(A;(u)) \xrightarrow{\det} \operatorname{Im}\det$ *is surjective* (*$R^{\bullet}$ means the multiplicative group of units of the ring R*),

(*b*) $U_2^{-1}(A;(u)) \to KU_1^{-1}(A;(u))$ *is surjective,*

then $KSp_1(A_0;(u)\cap A_0) \to W_1^{-1}(A;(u))$ *is surjective, where we write* KSp_1 *instead of* KU_1^{-1} *in the full sympletic case. Hence, when* $KSp_1(A_0;(u)\cap A_0)=0$, *then* $W_1^{-1}(A;(u))=0$. *In the latter case, if also* $K_1(A)\to K_1(A/(u))$ *is surjective, then* $TW_1(A;(u))=0$. *In this theorem,* $KU_1^{-1}(A;(u))$, $W_1^{-1}(A;(u))$, $KSp_1(A_0;(u)\cap A_0)$ *denote the relative groups* (*See* [5, *Chapter II, Section* 6]).

Proof. (1) By making use of (*a*), we can modify (*b*) to the surjection $Sp_2(A_0;(u)\cap A_0)\to W_1^{-1}(A;(u))$ as follows. Take $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in U_2^{-1}(A;(u))$, and suppose $\det\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)^{-1}=\det H(x)$ for some $x\in(1+(u))^{\bullet}$. The matrix $\sigma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)H(x)\in U_2^{-1}(A;(u))$ and has determinant 1. It can be shown then σ lies in $Sp_2(A_0;(u)\cap A_0)$ [8, Chapter I, (8.6)]. But σ and $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ both represent the same element in $W_1^{-1}(A;(u))$. From this, the first assertion follows. The second assertion is now immediate.

(2) Consider the exact sequence

$$1\to U^{-1}(A;(u))\to TU(A;(u))\to TU(A/(u)).$$

Since the restriction $ETU(A;(u))\to ETU(A/(u))$ is surjective, we obtain an exact sequence

$$KU_1^{-1}(A;(u))\to KTU_1(A;(u))\to KTU_1(A/(u)).$$

Suppose $K_1(A)\to K_1(A/(u))$ is surjective, then we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
K_1(A;(u)) & \longrightarrow & K_1(A) & \longrightarrow & K_1(A/(u)) & \longrightarrow & 0\\
{\scriptstyle H}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle H}\downarrow & & \\
KU_1^{-1}(A;(u)) & \to & KTU_1(A;(u)) & \to & KTU_1(A/(u)), & &
\end{array}$$

which gives rise to an exact sequence of cokernels

$$W_1^{-1}(A;(u))\to TW_1(A;(u))\to TW_1(A/u)).$$

By Proposition (5.4), $TW_1(A/(u))=0$, so $TW_1(A;(u))=0$ if $W_1^{-1}(A;(u))=0$.

THEOREM 5.6. *Suppose* $K_1(A)\to K_1(A/(u))$ *is surjective, then we have an exact sequence*

$$W_1{}^1(A;(u)) \xrightarrow{\varphi} W_1{}^1(A) \xrightarrow{\psi} W_1{}^1(A/(u))$$

with $\operatorname{Im} \varphi \subset \operatorname{Im} \Phi$ (*see Remark* 5.3). *The homomorphism* ψ *is induced from the natural epimorphism* $A \to A/(u)$. *Hence, when* $TW_1(A; (u)) = 0$, ψ *is a monomorphism.*

Proof. Consider the exact sequence

$$1 \longrightarrow U^1(A; (u)) \longrightarrow U^1(A) \xrightarrow{\psi} U^1(A/(u)).$$

Since the restriction $EU^1(A) \to EU^1(A/(u))$ is surjective, we obtain an exact sequence

$$KU_1^1(A; (u)) \xrightarrow{\varphi} KU_1^1(A) \xrightarrow{\psi} KU_1^1(A/(u)).$$

Suppose $K_1(A) \to K_1(A/(u))$ is surjective, then the same argument as in the proof of Theorem (5.5) gives the exact sequence we want. Clearly $\operatorname{Im} \varphi \subset \operatorname{Im} \Phi$.

We now return to our group ring. For the rest of this paper, π denotes a cyclic group of *odd* order m. We shall write $\mathbf{Z}[\tau]$ instead of $\mathbf{Z}\pi^*$, as explained in Section 4.

PROPOSITION 5.7. *Let* $u = \tau - \tau^{-1}$. *Then* u *is a nonzero-divisor in* $\mathbf{Z}[\tau]$ *satisfying* $u + \bar{u} = 0$, *and* $(u) = (\tau - 1)$, *and the homomorphism* $j_2 : \mathbf{Z}[\tau]/(u) \to \mathbf{Z}/m\mathbf{Z}$ *is an isomorphism.*

Proof. Clearly $\bar{u} = -u$. Since m is odd, $(\tau^2 - 1)/(\tau - 1)$ is a unit in $Z[\tau]$. Because $u = ((\tau^2 - 1)/(\tau - 1))\ (\tau - 1)\ 1/\tau$, and $\tau - 1$ is a nonzero-divisor, so u is a nonzero-divisor and $(u) = (\tau - 1)$. The epimorphism $j_2 : \mathbf{Z}[\tau] \to \mathbf{Z}/m\mathbf{Z}$ has kernel $i_1(t - 1) = (\tau - 1) = (u)$, so the last assertion follows.

THEOREM 5.8. *Let* $A = \mathbf{Z}[\tau]$, $u = \tau - \tau^{-1}$. *Then all conditions in Theorem* 5.5 *are satisfied, namely*

(*a*) A *satisfies the "evenness conditions"*,

(*b*) $K_1(A) \to K_1(A/(u))$ *is surjective,*

(*c*) $U_2^{-1}(A; (u)) \to KU_1^{-1}(A; (u))$ *is surjective,*

(*d*) $KSp_1(A_0\,; (u) \cap A_0) = 0$,

(*e*) *the composite* $(1 + (u))^{\bullet} \xrightarrow{H} KU_1^{-1}(A; (u)) \xrightarrow{\det} \operatorname{Im} \det$ *is surjective.*

Proof. (a) It has been remarked in Section 4 that A satisfies the "evenness condition" if m is odd.

(b) The homomorphism $\det : K_1(A/(u)) \to (\mathbf{Z}/m\mathbf{Z})^{\bullet}$ is an isomorphism since $A/(u)$ is isomorphic to $\mathbf{Z}/m\mathbf{Z}$ by Proposition (5.7). A unit of $\mathbf{Z}/m\mathbf{Z}$ is any

$n + m\mathbf{Z}$ with n, m relatively prime. Then $(\tau^n - 1)/(\tau - 1)$ is a unit in A, so that it represents an element of $K_1(A)$ which goes to $n + m\mathbf{Z}$. Hence the homomorphism $K_1(A) \to K_1(A/(u))$ is surjective.

(c) We have $A = \mathbf{Z}[\tau] \subset \prod_{d|m, d\neq 1} \mathbf{Q}[\zeta_d]$, where ζ_d is the primitive dth root of unity, so A is a commutative order in a semi-simple $\mathbf{Q}$-algebra. This guarantees the surjectivity of $U_2(A; (u)) \to KU_1^{-1}(A; (u))$ by stability theorems [5, Chapter IV, Section 3].

(d) We have $A_0 = \mathbf{Z}[\tau]_0 \subset \prod_{d|m, d\neq 1} \mathbf{Q}[\zeta_d + \zeta_d^{-1}]$, so A_0 is a commutative order in a semisimple $\mathbf{Q}$-algebra with no totally imaginary factors. This guarantees that $KSp_1(A_0\,; (u) \cap A_0) = 0$ [8, Chapter I, (8.11)] (or see [5]).

(e) Suppose $\det \sigma = x$, then x is of norm one, so that $x = \pm\tau^i$ for some i (to be proved below as Proposition (5.10)). By (c) we may assume $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a 2×2-matrix. We claim that x cannot be equal to $-\tau^i$. Suppose it is, then $ad - bc = -\tau^i$. Since $a\bar{d} - b\bar{c} = 1$ and $z \equiv \bar{z} \bmod (u)$ for all $z \in \mathbf{Z}[\tau]$, we get $-\tau^i \equiv 1 \bmod (u)$. Clearly $\tau \equiv 1 \bmod(u)$, so $\tau^i \equiv 1 \bmod(u)$, which implies $2\tau^i \equiv 0 \bmod(u)$, and so $2 \equiv 0 \bmod(u)$, contradicting the fact that $\mathrm{char}(A/(u)) = \mathrm{char}(\mathbf{Z}/m\mathbf{Z}) \neq 2$. Therefore $\det \sigma = \tau^i$ for some i. Put $y = \tau^{i/2} = \tau^{i(m+1)/2}$, then $y = 1 + (\tau^{i/2} - 1)$ is in $(1 + (u))^{\bullet}$ and $\det H(y) = \tau^i = x$.

Corollary 5.9. *The homomorphism* $j_2 : W_1^{\,1}(\mathbf{Z}[\tau], A^*) \to WO_1(\mathbf{Z}/m\mathbf{Z})$ *is a monomorphism.*

Proof. This is an immediate consequence of Theorem 5.5, Theorem 5.6 and Theorem 5.8.

Proposition 5.10. *A unit of norm one in* $\mathbf{Z}[\tau]$ *is of the form* $\pm\tau^i$ *for some* i.

Proof. Suffices to show that every unit of norm one in $\mathbf{Z}[\tau]$ lifts to an unit of norm one in $\mathbf{Z}\pi$, because then we can use Higman's Theorem [6, Theorem 3] in $\mathbf{Z}\pi$ and go down to $\mathbf{Z}[\tau]$ again. Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{Z}\pi & \hookrightarrow & \mathbf{Z} \times \prod\limits_{\substack{d|m \\ d\neq 1}} \mathbf{Z}[\zeta_d] \\ \Big\downarrow{\scriptstyle i_1} & & \Big\downarrow{\scriptstyle p} \\ \mathbf{Z}[\tau] & \hookrightarrow & \prod\limits_{\substack{d|m \\ d\neq 1}} \mathbf{Z}[\zeta_d] \end{array}$$

where p is the projection map. Given $v \in \mathbf{Z}[\tau]$ such that $v\bar{v} = 1$, first take $u \in \mathbf{Z}\pi$ such that $i_1(u) = v$. The projection of u on each $\mathbf{Z}[\zeta_d]$ is of norm one, since that of v is. It remains to show that the projection of u on $\mathbf{Z}$, which is

$i_2(u)$, can also be made to have norm one, that is, want to arrange $i_2(u) = \pm 1$. Consider the Commutative diagram

$$\begin{tikzcd} & \mathbf{Z}[\tau] \arrow[dr] & \\ \mathbf{Z}\pi \arrow[ur, "i_1"] \arrow[rr] \arrow[d, "i_2"'] & & \mathbf{Z}[\zeta_m] \arrow[d] \\ \mathbf{Z} \arrow[rr] & & \mathbf{Z}/m\mathbf{Z} \end{tikzcd}$$

$i_1(u)$ goes to ± 1 in $\mathbf{Z}/m\mathbf{Z}$ since elements of norm one in $\mathbf{Z}[\zeta_m]$ can only be of the form $\pm\zeta_m{}^i$ for some i. Hence $i_2(u)$ is equal to $\pm 1 + am$ for some $a \in \mathbf{Z}$. Now $i_1(u - a\Sigma) = i_1(u) = v$ and $i_2(u - a\Sigma) = \pm 1$. So $u - a\Sigma$ does the job.

6. Computation of $KU_1{}^1(\mathbf{Z}\pi, \Lambda)$

Let us rewrite the exact sequence (4.4) obtained in Section 4, but for convenience omit the Λ-decorations.

$$\begin{aligned} &KU_2{}^1(\mathbf{Z}[\tau]) \oplus KO_2(\mathbf{Z}) \\ &\xrightarrow{j_2 - j_1} KO_2(\mathbf{Z}/m\mathbf{Z}) \\ &\qquad\qquad \downarrow \partial \\ &KU_1{}^1(\mathbf{Z}\pi) \xrightarrow{(i_1, i_2)} KU_1{}^1(\mathbf{Z}[\tau]) \oplus KO_1(\mathbf{Z}) \xrightarrow{j_2 - j_1} KO_1(\mathbf{Z}/m\mathbf{Z}) \end{aligned} \tag{6.1}$$

PROPOSITION 6.2. *When $m = p^r$, p an odd prime and $r \geqslant 1$, the homomorphism $j_2 - j_1 : KU_2{}^1(\mathbf{Z}[\tau]) \oplus KO_2(\mathbf{Z}) \to KO_2(\mathbf{Z}/m\mathbf{Z})$ is surjective. In fact, the natural epimorphism $\mathbf{Z} \to \mathbf{Z}/p^r\mathbf{Z}$ induces an epimorphism $KO_2(\mathbf{Z}) \to KO_2(\mathbf{Z}/p^r\mathbf{Z})$.*

Proof. Consider the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{KSpin}_2(\mathbf{Z}) \arrow[r] \arrow[d] & KO_2(\mathbf{Z}) \arrow[r] \arrow[d] & \mu_2(\mathbf{Z}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{KSpin}_2(\mathbf{Z}/p^r\mathbf{Z}) \arrow[r] & KO_2(\mathbf{Z}/p^r\mathbf{Z}) \arrow[r] & \mu_2(\mathbf{Z}/p^r\mathbf{Z}) \arrow[r] & 0 \end{tikzcd}$$

Since $p^r \not\equiv 0 \bmod 4$, we have $K\mathrm{Spin}_2(\mathbf{Z}/p^r\mathbf{Z}) = 0$ [9, Chapter 3, (3.3)], so the leftmost vertical arrow is a surjection. The group $(\mathbf{Z}/p^r\mathbf{Z})^{\bullet}$ is cyclic of even order $p^{r-1}(p - 1)$, so it has exactly one element of order two, namely the element -1, so $\mu_2(\mathbf{Z}/p^r\mathbf{Z}) = \{\pm 1\}$. Since $\mu_2(\mathbf{Z}) = \{\pm 1\}$, the rightmost vertical arrow is also a surjection. Therefore the middle arrow $KO_2(\mathbf{Z}) \to KO_2(\mathbf{Z}/p^r\mathbf{Z})$ is a surjection.

PROPOSITION 6.3. *If $m = p^r$, p an odd prime and $r \geqslant 1$, then there is an exact sequence*

$$0 \longrightarrow W_1{}^1(\mathbf{Z}\pi) \xrightarrow{(i_1, i_2)} W_1{}^1(\mathbf{Z}[\tau]) \oplus WO_1(\mathbf{Z}) \xrightarrow{j_2 - j_1} WO_1(\mathbf{Z}/m\mathbf{Z}), \tag{6.4}$$

where we write WO_1 for $W_1{}^1$ in the orthogonal case.

Proof. By Proposition (6.2), the homomorphism ∂ in (6.1) is the zero map, and we can replace $KO_2(\mathbf{Z}/m\mathbf{Z})$ in (6.1) by the trivial group. Consider the following commutative diagram with exact rows, which gives rise to the (dotted) exact sequence of kernels and cokernels,

$$\begin{tikzcd}
 & & \ker 1 \arrow[r, dashed] \arrow[d] & \ker 2 \arrow[d] \arrow[dddll, dashed, "\delta"] & \\
 & K_1(\mathbf{Z}\pi) \arrow[r] \arrow[d, "H"'] & K_1(\mathbf{Z}[\tau]) \oplus K_1(\mathbf{Z}) \arrow[r] \arrow[d, "H"'] & K_1(\mathbf{Z}/m\mathbf{Z}) \arrow[r] \arrow[d, "H"'] & 0 \\
0 \arrow[r] & KU_1{}^1(\mathbf{Z}\pi) \arrow[r] \arrow[d] & KU_1{}^1(\mathbf{Z}[\tau]) \oplus KO_1(\mathbf{Z}) \arrow[r] \arrow[d] & KO_1(\mathbf{Z}/m\mathbf{Z}) \arrow[d] & \\
 & W_1{}^1(\mathbf{Z}\pi) \arrow[r, dashed] & W_1{}^1(\mathbf{Z}[\tau]) \oplus WO_1(\mathbf{Z}) \arrow[r, dashed] & WO_1(\mathbf{Z}/m\mathbf{Z}) &
\end{tikzcd}$$

We claim that $j_2 - j_1 : \ker 1 \to \ker 2$ is surjective, so that δ is the zero map. Now $\ker 2 = \ker H = (\mathbf{Z}/m\mathbf{Z})^{\bullet 2}$ [8, Chapter I, (3.5)]. Given $u^2 \in \ker 2$, where u is a unit in $\mathbf{Z}/m\mathbf{Z}$, we can take $w_0 \in K_1(Z[\tau])$ such that $j_2(w_0) = u$. Put $w = w_0\overline{w_0}$, then $j_2(w) = u^2$ and $H(w) = H(w_0\overline{w}_0) = 0$ in $KU_1{}^1(\mathbf{Z}[\tau])$ [8, Chapter I, (3.5)]. Hence $w \in \ker 1$. In the (dotted) exact sequence we can then replace ker2 by the trivial group and get (6.4).

At this point we have to investigate what $KO_1(\mathbf{Z})$ is. To achieve this end, we state a result of Bass on the orthogonal groups [4]. Notations and proof are to be found in [4].

THEOREM 6.5 (Bass). *Suppose Max (A) is a noetherian space of dimension 1, then the following sequences*

$$SK_1(A) \xrightarrow{H} KO_1(A) \xrightarrow{(\mathrm{SN}, \deg)} \mathrm{Discr}(A) \oplus Z_2(A) \to 0 \tag{6.6}$$

$$K_1(A) \xrightarrow{H} KO_1(A) \xrightarrow{(\mathrm{SN}, \deg)} {}_2\mathrm{Pic}(A) \oplus Z_2(A) \to 0 \tag{6.7}$$

are exact. There is an exact sequence

$$0 \to A^{\bullet}/A^{\bullet 2} \to \operatorname{Discr}(A) \to {}_2\operatorname{Pic}(A) \to 0. \tag{6.8}$$

$Z_2(A)$ *is isomorphic to the group* Γ *of all matrices* $\binom{1-e \;\; e}{e \;\; 1-e} \in O_2(A)$ *where e is an idempotent in* A, *and can also be described as the group of all locally constant functions* $\operatorname{Spec}(A) \to \mathbf{Z}/2\mathbf{Z}$. *If* 2 *is not a zero-divisor in* A, *then* det *maps* Γ *isomorphically onto the group of units* $1 - 2e$, *e an idempotent in* A.

PROPOSITION 6.9. *The homomorphism* $(\mathrm{SN}, \deg) : KO_1(\mathbf{Z}) \to \{\pm 1\} \oplus \langle w_1 \rangle$ *is an isomorphism. The homomorphism* $\deg : WO_1(\mathbf{Z}) \to \langle w_1 \rangle$ *is an isomorphism. Here* w_1 *is the matrix* $\binom{0 \;\; 1}{1 \;\; 0}$.

Proof. We have $\dim \operatorname{Max}(\mathbf{Z}) \leqslant 1$, $SK_1(\mathbf{Z}) = 0$ and $\operatorname{Pic}(\mathbf{Z}) = 0$ [2]. Also we have $\mathbf{Z}^{\bullet}/\mathbf{Z}^{\bullet 2} = \{\pm 1\}$ and $Z_2(\mathbf{Z}) = \langle w_1 \rangle$. By (6.6) and (6.8), the homomorphism $(\mathrm{SN}, \deg) : KO_1(\mathbf{Z}) \to \{\pm 1\} \oplus \langle w_1 \rangle$ is an isomorphism. Because $WO_1(\mathbf{Z})$ is the quotient $KO_1(\mathbf{Z})/H(K_1(\mathbf{Z}))$, by (6.7) the homomorphism $\deg : WO_1(\mathbf{Z}) \to \langle w_1 \rangle$ is an isomorphism.

THEOREM 6.10. *If* $m = p^r$, *p an odd prime and* $r \geqslant 1$, *then* $W_1{}^1(\mathbf{Z}\pi, \Lambda)$ *is a group of order two, generated by* w_1. *In particular,* $L_1(\pi) = 0$.

Proof. In the exact sequence (6.4), the restriction of $j_2 - j_1$ to $W_1{}^1(\mathbf{Z}[\tau])$ is j_2, which is injective by Corollary (5.9). The restriction of $j_2 - j_1$ to $WO_1(\mathbf{Z})$ is $-j_1$, which has image contained in $\operatorname{Im} j_2$ since $WO_1(\mathbf{Z})$ is generated by w_1 by Proposition (6.9). Hence $\operatorname{Im}(j_2 - j_1 \mid W_1{}^1(\mathbf{Z}[\tau])) = \operatorname{Im}(j_2 - j_1)$ and $\ker(j_2 - j_1 \mid W_1{}^1(\mathbf{Z}[\tau])) = 0$. This implies $\ker(j_2 - j_1) \simeq WO_1(\mathbf{Z})$, a group of order two, generated by w_1. Hence $W_1{}^1(\mathbf{Z}\pi, \Lambda) \simeq \operatorname{Im}(i_1, i_2) = \ker(j_2 - j_1)$ is a group of order two, generated by w_1.

It is no easy matter to determine the group $KU_1{}^1(\mathbf{Z}\pi, \Lambda)$, even in this case. We know there is an exact sequence

$$(\mathbf{Z}\pi)^{\bullet} \xrightarrow{H} KU_1{}^1(\mathbf{Z}\pi, \Lambda) \longrightarrow \langle w_1 \rangle \longrightarrow 0 \tag{6.11}$$

because the homomorphism $\det : K_1(\mathbf{Z}\pi) \to (\mathbf{Z}\pi)^{\bullet}$ is an isomorphism [2, Chapter XI, (7.3)]. However the image of H is not explicitly known. To conclude this paper, we shall merely try to give an exponent for the group $KU_1{}^1(\mathbf{Z}\pi, \Lambda)$ in this case. First observe that (6.11) is split exact by sending w_1 to w_1 in $KU_1{}^1(\mathbf{Z}\pi, \Lambda)$. Hence $KU_1{}^1(\mathbf{Z}\pi, \Lambda)$ is the direct sum of a group of order two and the group $\operatorname{Im} H$, the latter being isomorphic to $(\mathbf{Z}\pi)^{\bullet}/\ker H$.

LEMMA 6.12. *The group* $\operatorname{Im} H$ *has exponent* $2m$.

Proof. Since $\{x\bar{x} \mid x \in (\mathbf{Z}\pi)^{\bullet}\} \subset \ker H \subset \{x \in (\mathbf{Z}\pi)^{\bullet} \mid x = \bar{x}\}$ [8, Chapter I, (3.5)], there is an exact sequence

$$\frac{\{x \in (\mathbf{Z}\pi)^{\bullet} \mid x = \bar{x}\}}{\{x\bar{x} \mid x \in (\mathbf{Z}\pi)^{\bullet}\}} \to \frac{(\mathbf{Z}\pi)^{\bullet}}{\ker H} \to \frac{(\mathbf{Z}\pi)^{\bullet}}{\{x \in (\mathbf{Z}\pi)^{\bullet} \mid x = \bar{x}\}} \to 1.$$

The group on the left clearly has exponent two. We claim that the group on the left has exponent m, which would prove the assertion. Consider the homomorphism $\varphi : (\mathbf{Z}\pi)^{\bullet} \to (\mathbf{Z}\pi)^{\bullet}$ defined by $\varphi(x) = x\bar{x}^{-1}$, with kernel $\{x \in (\mathbf{Z}\pi)^{\bullet} \mid x = \bar{x}\}$. $\operatorname{Im}\varphi = \{x\bar{x}^{-1} \mid x \in (\mathbf{Z}\pi)^{\bullet}\}$ is contained in $\pm\pi$ since $x\bar{x}^{-1}$ has norm one, and hence of finite order, hence in $\pm\pi$ [6, Theorem 3]. In fact $\operatorname{Im}\varphi \subset \pi$. Otherwise there is a unit, say x, such that $x = -g\bar{x}$ for some some $g \in \pi$. Applying the augmentation map $i_2 : \mathbf{Z}\pi \to \mathbf{Z}$, we obtain $i_2(x) = -i_2(\bar{x}) = -i_2(x)$, so that $i_2(x) = 0$, contradicting the fact that x is a unit. Thus the group on the right, being isomorphic to $\operatorname{Im}\varphi$, has exponent m.

COROLLARY 6.13. *If $m = p^r$, p an odd prime and $r \geqslant 1$, then $KU_1^{\,1}(\mathbf{Z}\pi, \Lambda)$ has exponent $2m$.*

Proof. Clear from above.

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